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From Supersymmetry to Quantum Commutativity*

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Let (H, R) be a quasitriangular Hopf algebra acting on an algebra A. We study a concept of A being quantum commutative with respect to (H, R). Superalgebras which are graded commutative (called sometimes commutative superalgebras) are shown to be examples of such an action. There is an analogous notion of quantum commutativity for comodule algebra. The quantum plane $C_q[x, y]$ is an example, both under the coaction of quantum 2×2 matrices, and also in a more novel way at q a root of unity. If H is a cocommutative finite dimensional Hopf algebra and $(D(H), R)$ its Drinfeld double we show that H is quantum commutative with respect to $(D(H), R)$. We discuss further examples of such actions and coactions, and show that the category ${}_{A \# H}\mathrm{Mod}$ resembles (for such actions) the category of modules over commutative rings.

INTRODUCTION

This paper introduces the concept of "*commutative actions*" of quasitriangular Hopf algebras, or "*commutative coactions*" of their "dual," the so-called braided Hopf algebras. This notion encompasses commutativity of algebras and superalgebras on one hand and describes the quantum planes and superplanes on the other.

Specifically, let (H, R) be a quasitriangular Hopf algebra over a field k (with $R = \sum R^{(1)} \otimes R^{(2)} \in H \otimes H$) acting on an algebra A. We way that A is "quantum commutative" with respect to the action of (H, R) if the following is satisfied:

$$ab = \sum (R^{(2)} \cdot b)(R^{(1)} \cdot a), \qquad \text{all} \quad a, b \in A \tag{*}$$

(we shall sometimes say that A is H-commutative if there is no danger of ambiguity).

The isomorphism $\Psi : V \otimes W \to W \otimes V$ defined by

$$\Psi : v \otimes w \to \sum (R^{(2)} \cdot w) \otimes (R^{(1)} \cdot v),$$

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where V and W are H-modules is of essential importance in the study of the representation theory associated to (H, R), and lies in the basis of the connection between quantum Yang–Baxter equations and the theory of braids and knots [Maj, Ma]. It is thus natural to study algebras acted upon by (H, R) whose multiplication map is invariant under this correspondence.

When H is a cocommutative Hopf algebra, $(H, 1 \otimes 1)$ is a (trivial) quasitriangular Hopf algebra and then $\Psi = \tau$, the twist map, and A is H-commutative if and only if A is a commutative algebra. Thus our view is that quantum commutative algebras are to quasitriangular Hopf algebras what commutative algebras are to cocommutative Hopf algebras. Indeed, we show that their module categories have similar properties [Theorem 2.5.]

It should be noted that like the twist map, the correspondence Ψ sometimes satisfies a "symmetry" condition, namely,

$$\Psi^2 = \mathrm{id}.$$

Although for truly quasitriangular Hopf algebras, Ψ is usually not a symmetry, and then only partial results can be expected regarding the notion of quantum-commutativity, it is certainly worthwhile studying this notion for the following reasons:

(1) When (H, R) is triangular then $\Psi^2 = \mathrm{id}$ and general results are obtained.

(2) Even in the quasitriangular case we give examples and motivating results.

(3) Studying this "symmetric" notion leads to a more general "quasisymmetric" definition of quantum-commutativity, such as developed in [Maj′, Maj″]. (See also [G], [B].)

Instead of considering quasitriangular Hopf algebra actions one may consider a "dual" notion, that of braided bialgebras (Hopf algebras) $(H, \langle\ |\ \rangle$ [LT, Maj] and their coactions.

Let $(H, \langle\ |\ \rangle)$ be a braided bialgebra and let A be a left H-comodule algebra (with coaction, $\rho(a) = \sum a_{(1)} \otimes a_{(2)} \in H \otimes A$) then A is "quantum commutative" with respect to the coaction of $(H, \langle\ |\ \rangle)$ if

$$ab = \sum \langle a_{(1)} \mid b_{(1)} \rangle\, b_{(2)} a_{(2)}, \qquad \text{for all} \quad a, b \in A. \qquad (*')$$

The categorical point of view of our definitions is a specialization of [Ma]. Namely, a "quantum commutative" algebra in a braided monoidal category, is an object A, with "multiplication" $\mu : A \otimes A \to A$ in the category, such that the following diagram commutes:

$$
\begin{tikzcd}
A\otimes A \arrow[rr, "\mu"] \arrow[dr, "\Psi_{A,A}"'] & & A \\
& A\otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

where $\Psi_{V,W}: V\otimes W \to W\otimes V$ is the functorial isomorphism from $V\otimes W$ to $W\otimes V$. Hence the definitions of quantum commutativity apply on one hand to the category of left (H, R)-modules with the functorial isomorphism given by $\Psi(v\otimes w) = \sum R^{(2)}\cdot w\otimes R^{(1)}\cdot v$, and on the other to the category of left $(H, \langle\ |\ \rangle)$-comodules with the functorial isomorphism given by

$$\Psi(v\otimes w) = \sum \langle v_{(1)} \mid w_{(1)}\rangle\, w_{(2)}\otimes v_{(2)}.$$

The following are some natural examples:

(1) *Commutative superalgebras* (*Example* 1.4)

Let A be a Z_2-graded algebras A, with

$$ab = (-1)^{\deg a \deg b}\, ba,$$

for all homogeneous elements $a, b\in A$.

Such algebras can be viewed as H-module algebras, where $H = (kZ_2)^* \# kZ_2$ [Ha]. As we shall see, (Lemma 1.1) (H, R) with $R = p_{\bar 0}\otimes \bar 0 + p_{\bar 1}\otimes \bar 1$ (where $\{\bar 0, \bar 1\} = Z_2$), is a quasitriangular Hopf algebra, and A is quantum commutative with respect to (H, R) if and only if A is a commutative superalgebra.

Better yet, A can be viewed as an H-module algebra, where $H = kZ_2'$ [Maj″] (kZ_2' is kZ_2 as Hopf algebra but has a nonstandard triangular structure).

(2) *The quantum plane* (*Example* 1.8)

Let $\mathbf{C}_q[x, y]$, with $q^n = 1$, $xy = q^{-1}yz$, denote the quantum plane.

Let $H = D(kG)$, the Drinfeld double of kG, where $G = Z_n\times Z_n$. This is a quasitriangular Hopf algebra, which is shown to equal $(kG)^*\otimes kG$ for the above G, with $R = \sum_{g\in G} p_g\otimes g$. The quantum plane is shown to be quantum commutative with respect to an action of (H, R) on it.

Taking the dual view, let B arise from

$$R = \frac{1}{q}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

and $H = A(B)$, then $(H, \langle \mid \rangle_B)$ is a braided bialgebra and $\mathbf{C}_q[x, y]$ is an H-comodule algebra which is quantum commutative with respect to this coaction.

In Section 1 we discuss these examples in detail. We also show that another natural example of an H-commutative action is the following: Let H be a finite-dimensional cocommutative Hopf algebra and $(D(H), R)$ its Drinfeld double, then first we show that $D(H)$ acts on $k_\sigma[H]$ for any "twisting" σ on H, and moreover, when $\sigma = \varepsilon$, that is, $k_\sigma[H] = H$, then H is quantum-commutative with respect to this action (Theorem 1.5).

In Section 2 we study structural properties of quantum commutative algebras A acted upon by (H, R). Of particular importance will be the property that all H-stable left or right ideals of A are two sided and that $A^H \subset Z(A)$. We consider the category of left $A \# H$-modules which exhibits similar properties to its commutative analogue. In particular, we show in Theorem 2.5 that every left $A \# H$-module M is also a right A-module, and thus prove that $\otimes_A$ gives rise to a monoidal category whose objects are left $A \# H$-modules.

We prove that if (H, R) is triangular, and A and B are H-commutative then so is $A \otimes_k B$ (where the algebra structure of $A \otimes B$ is defined in the category of (H, R)-modules).

We end by defining the notion of an H-center. If A is any algebra acted upon by a quasitriangular (H, R), then A contains an H-commutative subalgeba which we term the H-center of A, $Z_H(A)$. Unlike $Z(A)$, the usual center of A, $Z_H(A)$ is always H-stable and when H is cocommutative (with $R = 1 \otimes 1$) it coincides with $Z(A)$ (which was shown to be H-stable by [C]).

0. Preliminaries

Our basic reference on Hopf algebras is Sweedler's book [S], on quasitriangular Hopf algebras is Majid's paper [Maj] and on braided Hopf algebras Larson and Towber's paper [LT].

Let H be a Hopf algebra over a field k with comultiplication $\Delta: H \to H \otimes H$, counit $\varepsilon: H \to k$ and antipode $S: H \to H$. We use the "sigma" notation for Δ, that is,

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)}, \qquad \text{for all} \quad h \in H.$$

Let A be an algebra with 1 over k, then H left acts on A (or A is a left H-module algebra) if:

(1) A is a unital left H-module.

(2) $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$, all $h \in H$, $a, b \in A$.

Note that it is not necessary to assume that $h \cdot 1 = \varepsilon(h)1$, it follows from the following identity proved in [C]:

$$(h \cdot a)b = \sum h_{(1)} \cdot (aS(h_{(2)})) \cdot b, \qquad \text{for all} \quad a, b \in A, h \in H. \tag{0.1}$$

When H acts on A there are two related algebras. The algebra of H-invariants, $A^H = \{a \in A \mid h \cdot a = \varepsilon(h)a, \text{ all } h \in H\}$, and the smash product $A \# H$. As a vector space $A \# H$ is $A \otimes H$ with multiplication defined by $(a \# h)(b \# g) = \sum a(h_{(1)} \cdot b) \# h_{(2)} g$, all $a, b \in A$, $h, g \in H$.

We may identify A with $A \# 1$ and H with $1 \# H$ in $A \# H$. We shall thus often write ah for $a \# h$. Moreover A is a left $A \# H$ module via $(a \# h) \cdot b = a(h \cdot b)$, all $a, b \in A$, $h \in A$. Thus left $A \# H$-submodules of A are just the H-stable left ideals of A. One can also define a right action of $A \# H$ on A [CFM] and the right $A \# H$-submodules are the H-stable right ideals of A.

A more general product construction of A and H is that of a crossed product $A \#_\sigma H$ [BCM, DT]. Let $\sigma : H \otimes H \to A$ be a k-bilinear map so that $A \otimes H$ becomes an associative algebra with identity element $1 \otimes 1$ under the product

$$(a \otimes h)(b \otimes g) = \sum (h_{(1)} \cdot b)\, \sigma(h_{(2)}, g_{(1)}) \otimes h_{(3)} g_{(2)}.$$

Then $A \#_\sigma H = A \otimes H$ as vector spaces and multiplication is given as above. It is usually assumed that σ is invertible.

A related, yet more complicated construction is that of a double crossproduct [Maj″, 3.2] of two Hopf algebras A and H, $A \bowtie H$, which is a Hopf algebra, as follows: Assume,

(1) $(A, \rightarrow)$ is a left H-module.

(2) $(H, \leftarrow)$ is a right A-module.

(3) $\Delta(h \rightarrow a) = \sum (h_{(1)} \rightarrow a_{(1)}) \otimes (h_{(2)} \rightarrow a_{(2)})$.

(4) $\Delta(h \leftarrow a) = \sum (h_{(1)} \leftarrow a_{(1)}) \otimes (h_{(2)} \leftarrow a_{(2)})$.

(5) $\varepsilon(h \rightarrow a) = \varepsilon(h \leftarrow a) = \varepsilon(h)\, \varepsilon(a)$. (That is, A, H are module coalgebras.) As a coalgebra, $A \bowtie H = A \otimes H$, and multiplication is defined by

$$(a \bowtie h)(a' \bowtie h') = \sum a(h_{(1)} \rightarrow a'_{(1)}) \bowtie (h_{(2)} \leftarrow a'_{(2)})h'.$$

For this multiplication to make $A \bowtie H$ into a Hopf algebra there exist several necessary and sufficient conditions. And then $A \bowtie 1$ and $1 \bowtie H$ are

Hopf subalgebras of $A \bowtie H$ isomorphic to A and H, respectively. These conditions are:

(6) $h \rightarrow 1 = \varepsilon(h)1$, $1 \leftarrow a = \varepsilon(a)1$.

(7) $h \rightarrow (aa') = \sum (h_{(1)} \rightarrow a_{(1)})((h_{(2)} \leftarrow a_{(2)}) \rightarrow a')$.

(8) $(h'h) \leftarrow a = \sum (h' \leftarrow (h_{(1)} \rightarrow a_{(1)}))(h_{(2)} \leftarrow a_{(2)})$.

(9) $\sum (h_{(2)} \leftarrow a_{(2)}) \otimes (h_{(1)} \leftarrow a_{(1)}) = \sum (h_{(1)} \leftarrow a_{(1)}) \otimes (h_{(2)} \rightarrow a_{(2)})$.

The antipode of this Hopf algebra is given by

$$S(a \bowtie h) = (1 \bowtie S_H(h))(S_A(a) \bowtie 1)$$

for all $a \in A$, $h \in H$.

Remark. It should be noted that if A acts on H trivially, that is $h \leftarrow a = \varepsilon(a)h$, all $h \in H$, $a \in A$, then (7) boils down to $h \rightarrow (aa') = \sum (h_{(1)} \rightarrow a)(h_{(2)} \rightarrow a')$, hence A is a left H-module algebra. The multiplication in $A \bowtie H$ is just the multiplication in $A \# H$. Hence when A acts on H trivially then $A \bowtie H = A \# H$.

A dual notion of an action is that of a coaction. An algebra A is said to be a left H-comodule algebra if there exists $\rho : A \rightarrow H \otimes A$ (denoted $\rho(a) = \sum a_{(1)} \otimes a_{(2)}$) satisfying: $(\mathrm{id} \otimes \rho) \circ \rho = (\Delta \otimes \mathrm{id}) \circ \rho$ and $(\varepsilon \otimes \mathrm{id}) \circ \rho = \mathrm{id}$ and ρ is multiplicative.

One important application of the double cross product construction occurs when H is a finite-dimensional Hopf algebras, for then H^* is also a Hopf algebra, and there exist natural actions of one on the other.

For each $h \in H$, $p \in H^*$ define a "natural" left action of H on H^* by

$$h \rightharpoonup p = \sum \langle p_{(2)}, h \rangle p_{(1)}.$$

One can also define a right action by

$$p \leftharpoonup h = \sum \langle p_{(1)}, h \rangle p_{(2)}.$$

This gives rise to another left action given by

$$h \rightharpoondown p = p \leftharpoonup S^{-1}(h) = \sum \langle S^{-1} p_{(1)}, h \rangle p_{(2)}.$$

Since $H \cong (H^*)^*$ these define actions of H^* on H as well.

The adjoint action of H on itself is given by

$$h \cdot_{\mathrm{ad}} g = \sum h_{(1)} g S(h_2)$$

for all h, $h \in H$.

Let $A = H^{*\mathrm{cop}}$, where A results from H^* by "twisting" the comultiplication in H^*, the antipode is S^{-1}, but multiplication an counit stay as in H^*.

Define a left H-module structure on A by

$$h \rightarrow p = \sum h_{(1)} \rightharpoonup p \leftharpoonup S^{-1}(h_{(2)}) \tag{0.2}$$

and a right A-module structure on H by

$$h \leftarrow p = \sum S^{-1}(p_{(1)}) \rightharpoonup h \leftharpoonup p_{(2)} \qquad \text{all} \quad h \in H, p \in A. \tag{0.3}$$

Then $A \bowtie H$ is a Hopf algebra, which is moreover quasitriangular (see definition below). It describes the so called Drinfeld's double of H and is denoted by $D(H)$ [Maj″, Example 4.6].

We define next a *quasitriangular Hopf algebra*, a notion introduced by Drinfeld [D], to be a pair (H, R), where H is a Hopf algebra over k and an invertible $R = \sum R^{(1)} \otimes R^{(2)} \in H \otimes H$ satisfying the following ($r = R$):

(1) $\sum \Delta(R^{(1)}) \otimes R^{(2)} = \sum R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$.

(2) $\sum R^{(1)} \otimes \Delta(R^{(2)}) = \sum R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$.

(3) $\Delta^{\mathrm{cop}}(h) = R\, \Delta(h)\, R^{-1}$, all $h \in H$ (where $\Delta^{\mathrm{cop}}(h) = \sum h_{(2)} \otimes h_{(1)}$).

It is a consequence of the above that $R^{-1} = \sum S(R^{(1)}) \otimes R^{(2)}$ and that $\sum \varepsilon(R^{(1)})\, R^{(2)} = 1$. (H, R) is called triangular if $R^{-1} = \sum R^{(2)} \otimes R^{(1)}$. Also, if (H, R) is quasitriangular then so is $(H^{\mathrm{cop}}, R^{\tau})$, where $R^{\tau} = \sum R^{(2)} \otimes R^{(1)}$.

In the previous construction of $D(H) = H^{*\mathrm{cop}} \bowtie H$, the element $R = \sum_i (\varepsilon \bowtie h_i)(h_i^* \bowtie 1)$, where $\{h_i\}$ and $\{h_i^*\}$ are dual bases of H and H^* respectively, makes $(D(H), R)$ into a quasitriangular Hopf algebra (see [Maj″, Sect 1.2] or [R] for detailed computations).

For any Hopf algebra H, the module structure of H on vector spaces V, W can be extended to their tensor product by

$$h \cdot (v \otimes w) = \sum (h_{(1)} \cdot v) \otimes (h_{(2)} \cdot w).$$

This makes $({}_H\mathrm{Mod}, \otimes_k, k)$ into a monoidal category.

If H is quasitriangular then this category is braided by Ψ [Maj], as recalled in the introduction.

1. Examples of Quantum Commutative Actions and Coactions

We first prove some facts about the connection between double cross products and smash products for cocommutative H.

LEMMA 1.1. *If H is a finite dimensional cocommutative Hopf algebra then*:

(a) $D(H)=H^{*\,\mathrm{cop}} \bowtie H = H^{*\,\mathrm{cop}} \# H$.

(b) *$(H^* \# H, R)$ is a quasitriangular Hopf algebra with $R=\sum h_i^* \otimes h_i$, where $\{h_i\}$ and $\{h_i^*\}$ are dual bases of H and H^*.*

Proof. (a) By the remark following the definition of a double cross product it is enough to show that $H^{*\,\mathrm{cop}}$ acts trivially on H. Well, if $h\in H$ and $p\in H^*$, then

$$\begin{aligned} h \leftarrow p &= \sum \langle S^{-1}(p_{(1)}), h_{(3)}\rangle\langle p_{(2)}, h_{(1)}\rangle\, h_{(2)} \\ &= \sum \langle S^{-1}(p_{(1)}), h_{(2)}\rangle\langle p_{(2)}, h_{(1)}\rangle\, h_{(3)} \\ &= \sum \langle p_{(2)}\, S^{-1}(p_{(1)}), h_{(1)}\rangle\, h_{(2)} = \varepsilon(p)h. \end{aligned}$$

(b) Since $(H^{*\,\mathrm{cop}} \# H)^{\mathrm{cop}} = H^* \# H^{\mathrm{cop}} = H^* \# H$, and since by (a) $H^{*\,\mathrm{cop}} \# H$ is a Hopf algebra, so is $H^* \# H$. Since R of $H^{*\,\mathrm{cop}} \# H$ is $\sum h_i \otimes h_i^*$, R of $H^* \# H$ is $R^\tau = \sum h_i^* \otimes h_i$.

The next lemma is a special case of the abstract characterization of double cross product in [Maj″, Proposition 3.12] but we give here a direct proof to a partial converse of the previous lemma.

LEMMA 1.2. *Let H be a cocommutative Hopf algebra (not necessarily finite-dimensional), A a Hopf algebra acted upon by H. If $A \# H$ is a Hopf algebra and the structure maps of A and H are the restrictions of the structure maps in $A \# H$ then $A \# H = A \bowtie H$ (where A acts on H trivially and the action of H on A gives A a structure of an H-module coalgebra).*

Proof. In $A \# H$ we have $h\cdot a=\sum h_{(1)}aS(h_{(2)})$, all $h\in H$, $a\in A$. Thus $\Delta(h\cdot a)=\sum h_{(1)}a_{(1)}S(h_{(4)})\otimes h_{(2)}a_{(2)}S(h_{(3)})$. However, since H is cocommutative this equals $=\sum h_{(1)}\cdot a_{(1)}\otimes h_{(2)}\cdot a_{(2)}$ hence condition (3) of the definition of double cross products is satisfied. Furthermore $\varepsilon(h\cdot a)=\varepsilon(h_{(1)}aS(h_{(2)}))=\varepsilon(h)\,\varepsilon(a)$.

The rest of the conditions are immediately satisfied, hence $A \# H = A \bowtie H$.

Remark. In Kostant's theorem, the smash product is a double cross product. That is, if H is a cocommutative Hopf algebra over an algebraically closed field k then

$$H = U(P) \# kG = U(P) \bowtie kG,$$

where P = primitive elements of H and G the group likes of H.

Let us specialize the construction $D(H)$ to $H = kG$, where G is a finite group. First, let $\{p_g\}$ be a basis of $(kG)^*$ dual to the basis $\{g\}_{g \in G}$ of kG. Specializing (0.2), $(kG)^*$ is a left kG-module coalgebra by

$$h \rightharpoonup p_g = p_{hgh^{-1}}, \qquad \text{all} \quad g, h \in G.$$

Now by Lemma 1.1, $((kG)^* \# kG, R)$ is a quasitriangular Hopf algebra with $R = \sum_{g \in G} p_g \otimes g$. If furthermore G is abelian then $(kG)^*$ is cocommutative, hence by Lemma 1.1, $H = (kG)^* \bowtie kG = (kG)^* \otimes kG$. Note that although this H is both commutative and cocommutative, (H, R) is a nontrivial quasitriangular Hopf algebra.

DEFINITION 1.3. Let (H, R) be a quasitriangular Hopf algebra acting on an algebra A. We say that: A is *quantum commutative* with respect to (H, R) if

$$ab = \sum (R^{(2)} \cdot b)(R^{(1)} \cdot a), \qquad \text{for all} \quad a, b \in A.$$

As mentioned in the introduction, commutative superalgebras are an example of a quantum commutative action. The following gives the details.

EXAMPLE 1.4. *Commutative Superalgebras.* Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$, a Z_2-graded algebra, hence a $(kZ_2)^*$-module algebra [CM]. A is a commutative superalgebra if $ab = (-1)^{\deg a \deg b}\, ba$, for all homogeneous elements of A. Define an action of kZ_2 on A as well by

$$g \cdot a = (-1)^{g \deg a}\, a, \qquad \text{all homogeneous } a, \text{ and } g \in \{\bar{0}, \bar{1}\}.$$

Thus A is a $(kZ_2)^* \# kZ_2$-module algebra. By the above, $((kZ_2)^* \# kZ_2, R)$, with $R = p_{\bar{0}} \otimes \bar{0} + p_{\bar{1}} \otimes \bar{1}$, is a quasitriangular Hopf algebra. We show that A is quantum commutative with respect to the above action. Let $a, b \in A_{\bar{1}}$, we show that $\sum (R^{(2)} \cdot b)(R^{(1)} \cdot a) = ab$.

$$\begin{aligned}\sum (R^{(2)} \cdot b)(R^{(1)} \cdot a) &= (p_{\bar{0}} \cdot b)(\bar{0} \cdot a) + (p_{\bar{1}} \cdot b)(\bar{1} \cdot a) \\ &= b(\bar{1} \cdot a) = b(-1)^{1 \deg a}\, a = -ba = ab.\end{aligned}$$

The proof for other homogeneous elements is similar, and thus the statement is true for all $a, b \in A$.

Another way of viewing commutative superalgebras is given by [Maj"]. Let $\operatorname{char}(k) \neq 2$, then the Z_2-graded modules are in 1–1 correspondence with the kZ'_2 modules, where kZ'_2 denotes kZ_2 with a nonstandard triangular structure, i.e., if $Z'_2 = \{1, g\}$ then kZ'_2 has a triangular structure defined by $R = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$. Each Z'_2-graded vector space is a kZ'_2-module by defining $g \cdot v = (-1)^{|v|} v$ for homogeneous v, and each kZ_2'-module is of course Z_2-graded. In this situation A is a commutative super algebra iff it is quantum commutative with respect to the action defined above.

In the following theorem we show that quantum-commutative actions occur in abundance.

Let H be a cocommutative finite-dimensional Hopf algebra, then $D(H) = H^{*\,\text{cop}} \# H$. We define an action on the crossed product $k \#_\sigma H$ (denoted by $k_\sigma[H]$, and called a twisted product).

Recall [BM] that the map: $\gamma : H \to k_\sigma[H]$ given by $\gamma(h) = 1 \#_\sigma h$ is invertible in the convolution algebra $\operatorname{Hom}(H, k_\sigma(H))$ with inverse μ given by $\mu(h) = \sum \sigma^{-1}(Sh_{(2)}, h_{(3)}) \#_\sigma Sh_{(1)}$.

Define now an action of $D(H)$ on $k_\sigma[H]$ by

$$(1 \bowtie h) \cdot (1 \#_\sigma l) = \sum \gamma(h_{(1)})(1 \#_\sigma l)\, \mu(h_{(2)})$$

$$(p \bowtie 1) \cdot (1 \#_\sigma l) = 1 \#_\sigma (l \leftharpoonup S(p))$$

for each $h, l \in H$, $p \in H^{*\,\text{cop}}$.

When there is no danger of ambiguity we identify $1 \bowtie h = h$, $p \bowtie 1 = p$, and $1 \#_\sigma l = 1 \# l$.

Note that when $\sigma = \varepsilon$, the action of H on itself is the usual adjoint action.

THEOREM 1.5. *Let H be a cocommutative finite-dimensional Hopf algebra. Let $(D(H), R)$ be its associated Drinfel'd double then*

(a) *Let $k_\sigma[H]$ be a twisted product with invertible cocycle σ, then $k_\sigma[H]$ is a $D(H)$-module algebra (acting as above).*

(b) *$k_\sigma[H]$ is quantum-commutative with respect to $(D(H), R)$.*

Proof. (a) Let us first show that for all $g, h, l, t \in H$

$$g \cdot (h \cdot (1 \# l)) = gh \cdot (1 \# l)$$

and

$$g \cdot ((1 \# l)(1 \# t)) = \sum (g_{(1)} \cdot (1 \# l))(g_{(2)} \cdot (1 \# t)).$$

Well, using $\sigma : H \otimes H \to k$, we have

$$\begin{aligned} &g \cdot (h \cdot (1 \# l)) \\ &\quad = \sum (1 \# g_{(1)})(1 \# h_{(1)})(1 \# l)(\sigma^{-1}(Sh_{(3)}, h_{(4)}) \# Sh_{(2)}) \\ &\qquad \times (\sigma^{-1}(Sg_{(3)}, g_{(4)}) \# Sg_{(2)}) \\ &\quad = \sum \sigma(g_{(1)}, h_{(1)})\, \sigma^{-1}(Sh_{(3)}, h_{(4)})\, \sigma^{-1}(Sg_{(3)}, g_{(4)}) \\ &\qquad \times \sigma(Sh_{(3)}, Sg_{(3)})(1 \# g_{(2)}h_{(2)})(1 \# l)(1 \# Sh_{(2)}Sg_{(2)}) \end{aligned}$$

While

$$gh\cdot(1 \# l)=\sum(1 \# g_{(1)}h_{(1)})(1 \# l)$$
$$\times(\sigma^{-1}(Sg_{(3)}h_{(3)}, g_{(4)}h_{(4)}) \# S(g_{(2)}h_{(2)})).$$

Since H is cocommutative, equality of both sides will occur if

$$\sum \sigma(g_{(1)}, h_{(1)})\,\sigma^{-1}(Sh_{(2)}, h_{(3)})\,\sigma^{-1}(Sg_{(2)}, g_{(3)})$$
$$\times\sigma(Sh_{(4)}, Sg_{(4)})\,\sigma(S(g_{(5)}h_{(5)}), g_{(6)}h_{(6)})$$
$$=\varepsilon(g)\,\varepsilon(h).$$

To prove it write $\sum(1 \# Sh_{(1)}Sg_{(1)})(1 \# g_{(2)}h_{(2)})$ in two different ways, (remembering cocommutativity of H).

First way,

$$\sum(1 \# Sh_{(1)}Sg_{(1)})(1 \# g_{(2)}h_{(2)})$$
$$=\sum \sigma(Sh_{(1)}Sg_{(1)}, g_{(2)}h_{(2)}) \# Sh_{(1)}Sg_{(1)}g_{(2)}h_{(2)}$$
$$=\sum \sigma(Sh_{(1)}Sg_{(1)}, g_{(2)}h_{(2)})\,\varepsilon(g_{(3)})\,\varepsilon(h_{(3)})$$
$$=\sum \sigma(Sh_{(1)}Sg_{(1)}, g_{(2)}h_{(2)}).$$

On the other hand, note that $1 \# gh=\sum \sigma^{-1}(g_{(1)}, h_{(1)})(1 \# g_{(2)})(1 \# h_{(2)})$, hence the second way is,

$$\sum(1 \# Sh_{(1)}Sg_{(1)})(1 \# g_{(2)}h_{(2)})$$
$$=\sum \sigma^{-1}(Sh_{(1)}, Sg_{(1)})\,\sigma^{-1}(g_{(2)}, h_{(2)})(1 \# Sh_{(3)})$$
$$\times(1 \# Sg_{(3)})(1 \# g_{(4)})(1 \# h_{(4)})$$
$$=\sum \sigma^{-1}(Sh_{(1)}, Sg_{(1)})\,\sigma^{-1}(g_{(2)}, h_{(2)})(1 \# Sh_{(3)})$$
$$\times(\sigma(Sg_{(3)}, g_{(4)}) \# Sg_{(3)}g_{(4)})(1 \# h_{(4)})$$
$$=\sum \sigma^{-1}(Sh_{(1)}, Sg_{(1)})\,\sigma^{-1}(g_{(2)}, h_{(2)})\,\sigma(Sg_{(3)}, g_{(4)})\,\varepsilon(g_{(5)})$$
$$\times(1 \# Sh_{(3)})(1 \# h_{(4)})$$
$$=\sum \sigma^{-1}(Sh_{(1)}, Sg_{(1)})\,\sigma^{-1}(g_{(2)}, h_{(2)})$$
$$\times\sigma(Sg_{(3)}, g_{(4)})\,\sigma(Sh_{(3)}, h_{(4)})\,\varepsilon(g_{(5)})\,\varepsilon(h_{(5)}).$$

From the above it follows that

$$\sum \sigma(Sh_{(1)}Sg_{(1)}, g_{(2)}h_{(2)})$$
$$= \sum \sigma^{-1}(Sh_{(1)}, Sg_{(1)})\, \sigma^{-1}(g_{(2)}, h_{(2)})\, \sigma(Sg_{(3)}, g_4)\, \sigma(Sh_{(3)}, h_{(4)}),$$

which yields

$$\sum \sigma(Sh_{(1)}Sg_{(1)}, g_{(2)}h_{(2)})\, \sigma(Sh_{(3)}, Sg_{(3)})$$
$$\times \sigma(g_{(4)}, h_{(4)})\, \sigma^{-1}(Sg_{(5)}, g_{(6)})\, \sigma^{-1}(Sh_{(5)}, h_{(6)})$$
$$= \varepsilon(g)\, \varepsilon(h)$$

as claimed.

The claim that

$$g \cdot ((1 \# l)(1 \# t)) = \sum (g_{(1)} \cdot (1 \# l))(g_{(2)} \cdot (1 \# t))$$

follows easily from the fact that γ and μ are convolution inverses of each other.

Next we show that for all $p, q \in H^{*\,\mathrm{cop}}$, $l, t \in H$, $pq \cdot (1 \# l) = p \cdot (q \cdot (1 \# l))$ and $p \cdot ((1 \# l)(1 \# t) = \sum (p_{(2)} \cdot (1 \# l))(p_{(1)} \cdot (1 \# t))$. The first identity is standard. The second follows from cocommutativity of H, as

$$p \cdot ((1 \# l)(1 \# t)) = p \cdot (\sigma(l_{(1)}, t_{(1)}) \# l_{(2)}t_{(2)})$$
$$= \sum \sigma(l_{(1)}, t_{(1)}) \langle Sp, l_{(2)}t_{(2)} \rangle \# l_{(3)}t_{(3)}$$
$$= \sum \sigma(l_{(1)}, t_{(1)}) \langle Sp_{(2)}, l_{(2)} \rangle \langle Sp_{(1)}, t_2 \rangle \# l_{(3)}t_{(3)},$$

while

$$\sum (p_{(2)} \cdot (1 \# l))(p_{(1)} \cdot (1 \# t))$$
$$= \sum (\langle Sp_{(2)}, l_{(1)} \rangle \# l_{(2)})(\langle Sp_{(1)}, t_{(1)} \rangle \# t_{(2)})$$
$$= \sum \sigma(l_{(2)}, t_{(2)}) \langle Sp_{(2)}, l_{(1)} \rangle \langle Sp_{(1)}, t_{(1)} \rangle \# l_{(3)}t_{(3)}.$$

Which equals the above

It remains to prove that

$$[(1 \bowtie g)(p \bowtie 1)] \cdot (1 \# l) = (1 \bowtie g) \cdot ((p \bowtie 1) \cdot (1 \# l))$$

for all $g, l \in H$, $p \in H^{*\,\mathrm{cop}}$.

$$(1 \bowtie g) \cdot [(p \bowtie 1) \cdot (1 \# l)]$$
$$= \sum (1 \bowtie g)(1 \# \langle Sp, l_{(1)} \rangle l_{(2)})$$
$$= \sum \langle Sp, l_{(1)} \rangle \sigma^{-1}(Sg_{(3)}, g_{(4)})(1 \# g_{(1)})(1 \# l_{(2)})(1 \# Sg_{(2)}).$$

On the other hand,

$$((1 \bowtie g)(p \bowtie 1)) \cdot (1 \# l)$$
$$= \sum ((g_{(1)} \rightarrow p) \bowtie g_{(2)}) \cdot (1 \# l)$$
$$= \sum \langle g_{(1)}, p_{(3)} \rangle \langle Sg_{(2)}, p_{(1)} \rangle \, p_{(2)} \cdot (g_{(3)} \cdot (1 \# l))$$
$$= \sum \langle g_{(1)}, p_{(3)} \rangle \langle Sg_{(2)}, p_{(1)} \rangle \, \sigma^{-1}$$
$$\times (Sg_{(4)}, g_{(5)}) \, p_2 \cdot ((1 \# g_{(3)})(1 \# l)(1 \# Sg_{(6)}))$$
$$= \sum \langle g_{(1)}, p_{(5)} \rangle \langle Sg_{(2)}, p_{(1)} \rangle \sigma^{-1}$$
$$\times (Sg_{(4)}, g_{(5)})((1 \# g_{(3)}) \leftharpoonup Sp_{(4)})((1 \# l) \leftharpoonup Sp_{(3)})$$
$$\times ((1 \# Sg_{(6)}) \leftharpoonup Sp_{(2)})$$
$$= \sum \langle g_{(1)}, p_{(5)} \rangle \langle Sg_{(2)}, p_{(1)} \rangle \, \sigma^{-1}(Sg_{(5)}, g_{(6)})$$
$$\times \langle Sp_{(4)}, g_{(3)} \rangle \langle Sp_{(3)}, l_{(1)} \rangle \langle Sp_{(2)}, Sg_{(6)} \rangle$$
$$\times (1 \# g_{(4)})(1 \# l_{(2)})(1 \# Sg_{(7)}).$$

Since H is cocomutative we obtain the above to equal

$$= \sum \sigma^{-1}(Sg_{(1)}, g_{(3)}) \langle Sp, l_{(1)} \rangle (1 \# g_{(1)})(1 \# l_{(2)})(1 \# Sg_{(4)}).$$

We have thus shown that $k_\sigma[H]$ is a $D(H)$-module algebra.

(b) We need to show that for all $a, b \in H$

$$ab = \sum (R^{(2)} \cdot b)(R^{(1)} \cdot a),$$

where $R = \sum (\varepsilon \bowtie h_i)(h_i^* \bowtie 1)$ and $\{h_i\}$ and $\{h_i^*\}$ are dual bases of H and H^*, respectively.

Let $a = (1 \#_\sigma k)$ and $b = (1 \#_\sigma l)$, then

$$\begin{aligned}\sum (R^{(2)} \cdot b)(R^{(1)} \cdot a) &= \sum ((h_i^* \bowtie 1) \cdot (1 \#_\sigma l))((\varepsilon \bowtie h_i) \cdot (1 \#_\sigma k)) \\ &= \sum (1 \#_\sigma (l \leftharpoonup Sh_i^*))(h_i \cdot (1 \#_\sigma k)) \\ &= \sum (1 \#_\sigma \langle Sh_i^*, l_{(1)} \rangle l_{(2)})((h_i \cdot (1 \#_\sigma k)) \\ &= \sum (1 \#_\sigma l_{(2)})(\langle h_i^*, Sl_{(1)} \rangle h_i \cdot (1 \#_\sigma k)).\end{aligned}$$

But $\sum_i \langle h_i^*, g \rangle h_i = g$ for all $g \in H$, hence the above equals

$$\begin{aligned}&= \sum (1 \#_\sigma l_{(2)})(Sl_{(1)} \cdot (1 \#_\sigma k) \\ &= \sum (1 \#_\sigma l_{(3)})(1 \#_\sigma Sl_{(2)})(1 \#_\sigma k)\, \mu(Sl_{(1)}) \\ &= (\sigma(l_{(4)}, Sl_{(2)}) \#_\sigma l_{(5)} Sl_{(3)})(1 \#_\sigma k)\, \mu(Sl_{(1)}).\end{aligned}$$

Since H is cocommutative, the above equals

$$\begin{aligned}&(1 \#_\sigma k)(\sigma(l_{(2)}, Sl_{(3)})\, \mu(Sl_{(1)}) \\ &(1 \#_\sigma k)(\sigma(l_{(4)}, Sl_{(5)})\, \sigma^{-1}(l_{(2)}, Sl_{(1)}) \#_\sigma l_{(3)}) \\ &(1 \#_\sigma k)(1 \#_\sigma l).\end{aligned}$$

EXAMPLE 1.6. Let V be an H-module and let $T(V) = \sum V^{\otimes n}$ be the tensor algebra. When H is cocommutative, the ideal I of $T(V)$ generated by $\{v \otimes w - w \otimes v\}$ is H-stable for then

$$\begin{aligned}&h \cdot (v \otimes w - w \otimes v) \\ &\quad = \sum (h_{(1)} \cdot v \otimes h_{(2)} \cdot w) - (h_{(1)} \cdot w \otimes h_{(2)} \cdot v) \\ &\quad = \sum h_{(1)} \cdot v \otimes h_{(2)} \cdot w - h_{(2)} \cdot w \otimes h_{(1)} \cdot v \in I.\end{aligned}$$

Hence $S(V) = T(V)/I$ is a commutative algebra acted upon by H.

Generalizing to a quasitriangular Hopf algebra (H, R), the ideal I of $T(V)$ generated by $\{v \otimes w - \sum (R^{(2)} \cdot w)(R^{(1)} \cdot v)\}$ is H-stable for

$$\begin{aligned}&h \cdot (v \otimes w - R^{(2)} \cdot w \otimes R^{(1)} \cdot v) \\ &\quad = \sum h_{(1)} \cdot v \otimes h_{(2)} \cdot w - h_{(1)} R^{(2)} \cdot w \otimes h_{(2)} R^{(1)} \cdot v \\ &\quad = \sum (h_{(1)} \cdot v \otimes h_{(2)} \cdot w - R^{(2)} h_{(2)} \cdot w \otimes R^{(1)} h_{(1)} \cdot v) \in I.\end{aligned}$$

Let $S_R(V) = T(V)/I$, then $S_R(V)$ is H-commutative.

This construction was emphasized by Manin in the triangular case, but a few special quasitriangular examples are well known. See [G].

We introduce now a dual notion to (H, R)-commutativity. We follow the definitions of [LT, Maj]. We use [LT] terminology since it is the most thorough treatment of the topic, although there exist other prior terms of the dual notion of quasitriangular.

Let H be a bialgebra with a bilinear pairing $\langle \mid \rangle : H \otimes H \to k$ satisfying:

(1) $\sum \langle a_{(1)} \mid b_{(1)} \rangle b_{(2)} a_{(2)} = \sum \langle a_{(2)} \mid b_{(2)} \rangle a_{(1)} b_{(1)}$.

(2) $\langle \mid \rangle$ is regular.

(3) $\langle a \mid bc \rangle = \sum \langle a_{(1)} \mid b \rangle \langle a_{(2)} \mid c \rangle$.

(4) $\langle ab \mid c \rangle = \sum \langle b \mid c_{(1)} \rangle \langle a \mid c_{(2)} \rangle$.

for all $a, b, c \in H$.

Then $(H, \langle \mid \rangle)$ is called a *braided bialgebra.*

DEFINITION 1.3′. Let $(H, \langle \mid \rangle)$ be a braided bialgebra, let A be an H-comodule algebra then we say that A is *quantum commutative* with respect to $(H, \langle \mid \rangle)$ if

$$ab = \sum \langle a_{(1)} \mid b_{(1)} \rangle b_{(2)} a_{(2)}$$

for all $a, b \in A$.

The following is the motivating example for braided bialgebras. Let V be a finite-dimensional vector space over k with basis $\{f^i\}_{i=1,\ldots,n}$. Let $B : V \otimes V \to V \otimes V$ be an operator whose matrix with respect to $\{f^i \otimes f^j\}$ (denoted by B as well) is indexed by the pairs (ij, kl) $i, j, k, l = 1, \ldots, n$, where B_{ij}^{kl} denotes the entry in B in the (ij) row, (k, l)th column. That is

$$B(f^i \otimes f^j) = \sum B_{ij}^{kl} f^k \otimes f^l. \tag{1}$$

Assume further that B satisfies the *braid condition*: $B_{12} B_{23} B_{12} = B_{23} B_{12} B_{23}$ (this takes place in $M_n(k)^{\otimes 3}$, where B_{ij} means B in the i, j position). This condition is equivalent to the so called matrix quantum Yang-Baxter equations. One forms now, $A(B)$, a bialgebra associated to B (usually denoted by $A(R)$). It arises from the free algebra generated by: $\{T_{ij} \mid 1 \leqslant i, j \leqslant n\}$ subject to the relations,

$$\sum_{I,J} T_{iI} T_{jJ} B_{IJ}^{mp} = \sum_{I,J} T_{Im} T_{Jp} B_{ij}^{IJ}. \tag{2}$$

Define next a bilinear pairing $\langle \mid \rangle_B : A(B) \otimes A(B) \to k$ by

$$\langle T_{ij} \mid T_{IJ} \rangle_B = B_{iI}^{Jj} \tag{3}$$

then $(A(B), \langle \mid \rangle_B)$ is a braided bialgebra [LT, Maj].

Now V becomes a left $A(B)$ comodule via $\rho : V \to A(B) \otimes V$, given by

$$\rho(f^i) = \sum_k T_{ik} \otimes f^k. \tag{4}$$

As in Example 1.6, let $S_B(V) = T(V)/I$, the tensor algebra of V modulo the relations

$$f^i \otimes f^j = B(f^i \otimes f^j).$$

Explicitly, using (1)

$$f^i f^j = \sum_{k,l} B_{ij}^{kl} f^k f^l. \tag{5}$$

Then we prove the analogue of Example 1.6. It has the analogous universal property.

PROPOSITION 1.7. *Let V be a finite-dimensional vector space over k. Let B be an operator: $V \otimes V \to V \otimes V$ satisfying the braid condition, then $S_B(V)$ is quantum-commutative with respect to the coaction of $(A(B), \langle \mid \rangle_B)$.*

Proof. First, $S_B(V)$ is an $A(B)$-comodule algebra. We prove it here for completeness (it is proved in [Maj, p. 55]). $T(V)$ is a comodule algebra by extending the coaction of ρ on V multiplicatively. We must show that $\rho(I) = 0$, that is, using (5), we must show

$$\rho(f^i f^j) = \sum B_{ij}^{kl} \rho(f^j f^l).$$

Well, by (4), the left hand side is

$$\begin{aligned} \rho(f^i)\,\rho(f^j) &= \sum_{m,p} T_{im} T_{jp} \otimes f^m f^p \\ &= \sum_{m,p,I,J} B_{mp}^{IJ} T_{im} T_{jp} \otimes f^I f^J \qquad \text{by (5)} \end{aligned}$$

while the right hand side,

$$\sum_{k,l} B_{ij}^{kl} \rho(f^k f^l) = \sum_{k,l,s,u} B_{ij}^{kl} T_{ks} T_{lu} \otimes f^s f^u.$$

This now equals the left hand side by (2).

Now we show that $S_B(V)$ is quantum commutative. We show it for $\{f^i\}$ first,

$$\begin{aligned}\sum \langle f^i_{(1)} \mid f^j_{(1)} \rangle_B f^j_{(2)} f^i_{(2)} &= \sum_{k,l} \langle T_{ik}, T_{jl} \rangle_B f^l f^k \\ &= \sum_{k,l} B^{lk}_{ij} f^l f^k = f^i f^j \qquad \text{(by (3))}.\end{aligned}$$

Using properties (3) and (4) of the definition of a braided bialgebra it is straightforward to show that

$$ab = \sum \langle a_{(1)} \mid b_{(1)} \rangle_B b_{(2)} a_{(2)}, \qquad \text{for all} \quad a, b \in S_B(V).$$

We apply the proposition to the quantum plane and the quantum superplane and show they are all examples of a quantum commutative coaction. Moreover, we show that just as for superalgebras, there exist quasitriagular Hopf algebras (H, R), (unrelated to $A(B)$) with respect to which the quantum plane and superplane are quantum commutative H-module algebras as well).

EXAMPLE 1.8. *The quantum plane* $C_q[x, y]$. $C_q[x, y]$ equals the free algebra $C\langle x, y\rangle$ modulo the relation $xy = q^{-1}yx$.

(a) $C_q[x,y]$ is quantum commutative with respect to a *coaction*: Let

$$R = \frac{1}{q}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

Let

$$B = \tau \circ R = \frac{1}{q}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad \text{(where } \tau = \text{twist map)}.$$

it is well known that B satisfies the braid conditions.

The normalization here for R is chosen carefully to avoid a non-trivial algebra in the construction of $S_B(V)$ [Ma].

Denote in the proposition $f^1 = x$, $f^2 = y$, and V the vector space with basis $\{x, y\}$, then it is immediate that $S_B(V) = C_q[x, y]$ and so by the proposition $C_q[x, y]$ is quantum commutative with respect to the coaction of $(A(B), \langle\ |\ \rangle_B)$. $A(B)$ is sometimes denoted by ${}_qM_2(\mathbf{C})$.

(b) The quantum plane with q a root of 1 is quantum commutative with respect to an *action* of (H, R).

Let $G = Z_n \times Z_n$ and let $H = (kG)^* \bowtie kG$, note that $H = (kG)^* \otimes kG$. (Since G is abelian), and recall $R = \sum_{(i,j) \in G} (i, j) \otimes p_{(i,j)}$ (see Lemma 1.2). Since $C_q[x, y]$ has the standard $Z \times Z$ grading according to degrees of x and y, and since $q^n = 1$ implies x^n, y^n belong to the center of $C_q[x, y]$ we have

$$C_q[x, y] = \bigoplus_{(i,j) \in G} C[x^n, y^n]\, x^i y^j.$$

That is, $C_q[x, y]$ is G-graded, and so by [CM] $(kG)^*$ acts on $C_q[x, y]$. Motivated by [Ha], define an action of kG on $C_q[x, y]$ as

$$(i, j) \cdot x = q^{-j} x$$
$$(i, j) \cdot y = q^i y$$

and extend this to $C_q[x, y]$. Now, we show H-commutativity,

$$\sum (R^{(2)} \cdot y)(R^{(1)} \cdot x) = \sum_{(i,j) \in G} (p_{(i,j)} \cdot y)((i, j) \cdot x).$$

But $y \in (C_q[x, y])_{(0,1)}$ and $x \in (C_q[x, y])_{(1,0)}$ hence this equals

$$y((0, 1) \cdot x) = yq^{-1}x = q^{-1}yx = xy.$$

Hence $C_q[x, y]$ is quantum-commutative with respect to (H, R).

EXAMPLE 1.9. *The quantum superplane* $C_q[\xi, \eta]$. $C_q[\xi, \eta]$ equals the free algebra $C\langle \xi, \eta)$ subject to: $\xi^2 = \eta^2 = 0$, $\xi\eta = -q\eta\xi$ (then it is also called the q-Grassman algebra).

(a) If we replace R by $-q^2 R$ in Example 1.8 then in an analogous manner $C_q[\xi, \eta] = S_{(-q^2 B)}(V)$ which is quantum-commutative with respect to the coaction of $(A(-q^2 B), \langle\ |\ \rangle_{-q^2 B})$.

(b) Take (H, R) as in Example 1.8 but define the action of kG by

$$(i, j) \cdot \xi = -q^{-j} \xi$$
$$(i, j) \cdot \eta = -q^i \eta$$

The grading is as in Example 1.8. Then $C_q[\xi, \eta]$ is quantum-commutative with respect to the action of (H, R).

2. Structural Properties

Let A be quantum commutative with respect to (H, R). In this section we study structural properties of A and of the category of left $A \# H$-modules, denoted by ${}_{A \# H}\mathrm{Mod}$.

First observe that since $\sum S(R^{(1)}) \otimes R^{(2)} = R^{-1}$, we have for all $a, b \in A$

$$ab = \sum (S(R^{(1)}) \cdot b)(R^{(2)} \cdot a). \tag{2.1}$$

Following Radford's definition [R] let $L = R_{(l)} = Sp_k\{R^{(1)}\}$, then L is a finite dimensional Hopf subalgebra of $H(R_{(r)}$ is defined analogously).

Proposition 2.2. (1) *Every L-stable left (right) ideal of A is two-sided.*

(2) $A^L \subset Z(A)$.

In particular, H-stable left (right) ideals of A are two sided and $A^H \subset Z(A)$.

(3) *If I is an H-stable ideal of A then its left and right annihilators in A coincide.*

Proof. (1) Let I be a left L-stable ideal of A, we prove it is a right ideal. Let $a \in I$, $b \in A$ then $ab = \sum (R^{(2)} \cdot b)(R^{(1)} \cdot a) \in I$ since $R^{(1)} \in L$ and I is L-stable. If I is a right L-stable ideal then for $a \in I$, $b \in I$, $ba = \sum (S(R^{(1)}) \cdot a)(R^{(2)} \cdot b)$ by (2.1), and hence $ba \in I$.

(2) Let $a \in A^L$, $b \in A$ then

$$ab = \sum (R^{(2)} \cdot b)(R^{(1)} \cdot a) = \sum (R^{(2)} \cdot b)(\varepsilon(R^{(1)})a)$$
$$= \left(\sum (\varepsilon(R^{(1)})\, R^{(2)}) \cdot b\right) a = ba.$$

That is, $a \in Z(A)$.

Let $T = R_{(r)}$ then Proposition 2.2 is satisfied with T replacing L, the proof is analogous.

(3) Let I be an H-stable ideal of A. By [C] its right and left annihilators (denoted by $r_A(I)$, $l_A(I)$ respectively) are H-stable.

Let $x \in l_A I)$, $y \in I$ then

$$yx = \sum (R^{(2)} \cdot x)(R^{(1)} \cdot y) \in l_A(I) \cdot I = 0.$$

Hence $x \in r_A(I)$ and vice versa. Thus $r_A(I) = l_A(I)$.

Remark. When $H = (kG)^* \# kG$, Proposition 2.2 implies that either G-graded or G-stable left (right) ideals are already two-sided, and that $A_1 \cup A^G \subset Z(A)$.

When H is any Hopf algebra acting on an algebra A, the algebra is H-simple if it has no nontrivial H-stable ideals, this usually does not imply that A is an irreducible left (or right) $A \# H$-module. In our situation however these notions coincide.

COROLLARY 2.3. *A is H-simple if and only if A is an irreducible left (or right) $A \# H$-module, and then A^H is a field.*

Proof. Since $A \# H$-submodules of A are the left (right) H-stable ideals, this is a corollary of Proposition 2.2. The fact that A^H is a field follows from the fact that $A^H \cong \mathrm{End}_{A \# H} A$ and Schur's lemma.

We can thus prove an analogue of [CFM, 3.4].

COROLLARY 2.4. *Let A be quantum-commutative with respect to a finite-dimensional (H, R), then the following are equivalent:*

(1) *$A \# H$ is simple and A has finite Goldie rank.*

(2) *$A \# H$ is simple Artinian.*

(3) *A is H-simple and $[A : A^H] = \dim_k H$.*

Proposition 2.2 rephrased is a statement about the $A \# H$-module A. The following is its generalization to any $A \# H$-module M.

When A is a commutative superalgebra and M is a graded left A-module (a left $A \# H$-module with $H = kZ_2'$, where kZ_2' was defined in Example 1.4) then it is a right A-module as well [Kos]. We generalize this to any (H, R) and any H-commutative A, and show that ${}_{A \# H}\mathrm{Mod}$ behaves "nicely."

From the representation theory of (H, R) we know [Maj] that the category of left H-modules is a monoidal quasitensorial category, $({}_H\mathrm{Mod}, \otimes_k, \Phi, \Psi, k, \mathrm{Hom}_k(M, N))$, where $\Psi : M \otimes_k N \to N \otimes_k M$ is given by $\Psi(m \otimes n) = \sum R^{(2)} \cdot n \otimes R^{(1)} \cdot m$ and $\Phi : (M_1 \otimes_k M_2) \otimes_k M_3 \to M_1 \otimes_k (M_2 \otimes_k M_3)$ is the standard vector-space isomorphism. We explicitly show:

THEOREM 2.5. *Let (H, R) be a quasitriangular Hopf algebra acting on A, and assume A is quantum-commutative with respect to (H, R) then:*

(1) *$({}_{A \# H}\mathrm{Mod}, \otimes_A, \Phi, A)$ is monoidal category.*

(2) *for any $M, N \in {}_{A \# H}\mathrm{Mod}$,*

$$\mathrm{Hom}_A(M_A, N_A) \in {}_{A \# H}\mathrm{Mod}$$

and

$$(\mathrm{Hom}_A(M_A, N_A))^H = \mathrm{Hom}_{A \# H}({}_AM, {}_AN).$$

Proof. (1) Let M be a left $A \# H$ module. Define a right action of A on M by

$$m \leftharpoonup a = \sum (R^{(2)} \cdot a)(R^{(1)} \cdot (m))$$

that is the right action $\tilde{\lambda} : M \otimes A \to M$ is given by $\tilde{\lambda} = \lambda \circ \Psi$ (where λ is the left A-module action and Ψ is defined as above). Since both λ and Ψ are H-module homomorphisms so is $\tilde{\lambda}$. Thus $h \cdot (m \leftharpoonup a) = \sum (h_1 \cdot m) \leftharpoonup (h_2 \cdot a)$. We postpone showing that $m \leftharpoonup ab = (m \leftharpoonup a) \leftharpoonup b$ to the next paragraph.

Furthermore, the pentagon and hexagon identities for Φ and Ψ together with the identities for H-commutativity, that is

$$\begin{tikzcd} A \otimes A \arrow[rr, "\mu"] \arrow[dr, "\Psi"'] & & A \\ & A \otimes A \arrow[ur, "\mu"'] & \end{tikzcd}$$

(where μ denotes the multiplication on A) make M into an A–A bimodule.

Explicitly, for all $a, b \in A$, $m \in M$

$$\begin{aligned} (a \cdot m) \leftharpoonup b &= \sum (R^{(2)} \cdot b)(R^{(1)} \cdot (am)) \\ &= \sum (R^{(2)} \cdot b)(R^{(1)}_{(1)} \cdot a)(R^{(1)}_{(2)} \cdot m) \\ &= \sum (R^{(2)} r^{(2)} \cdot b)(R^{(1)} \cdot a)(r^{(1)} \cdot m) \\ &= \sum a(r^{(2)} \cdot b)(r^{(1)} \cdot m) \\ &= a(m \leftharpoonup b). \end{aligned}$$

Now we are ready to show that $(m \leftharpoonup ab) = (m \leftharpoonup a) \leftharpoonup b$, well,

$$\begin{aligned} m \leftharpoonup ab &= \sum (R^{(2)} \cdot (ab))(R^{(1)} \cdot m) \\ &= \sum (R^{(2)}_{(1)} \cdot a)(R^{(2)}_{(2)} \cdot b)(R^{(1)} \cdot m) \\ &= \sum (r^{(2)} \cdot a)(R^{(2)} \cdot b)(R^{(1)} r^{(1)} \cdot m) \\ &= \sum (r^{(2)} \cdot a)((r^{(1)} \cdot m) \leftharpoonup b) \qquad \text{(by the above)} \\ &= \sum ((r^{(2)} \cdot a)(r^{(1)} \cdot m)) \leftharpoonup b = (m \leftharpoonup a) \leftharpoonup b. \end{aligned}$$

Due to the above, if $M, N \in {}_{A\,\#\,H}\mathrm{Mod}$, one can form $M \otimes_A N$ (instead of tensoring over k) and make it into a left $A \# H$-module as usual,

$$(a \# h)\cdot(m \otimes_A n) = \sum ah_{(1)}\cdot m \otimes_A h_{(2)}\cdot n \qquad \text{for all} \quad a\in A, h\in H, m\in M, n\in N.$$

The action of A on $M \otimes_A N$ is well defined since M, N are bimodules.

The action of H on $M \otimes_A N$ is well defined since $\tilde{\lambda}$ is an H-module homomorphism.

The standard function $\Phi_{M_1, M_2, M_3}: M_1 \otimes_A)(M_2 \otimes_A M_3) \to (M_1 \otimes_A M_2) \otimes_A M_3$, is easily seen to be an A-module isomorphism, since it is routine to check that $(m_1 \otimes m_2) \leftharpoonup a = m_1 \otimes (m_2 \leftharpoonup a)$.

Thus $({}_{A\,\#\,H}\mathrm{Mod}, \otimes_A, \Phi, A)$ is a monoidal category.

(2) Let $N, M \in {}_{A\,\#\,H}\mathrm{Mod}$, then by (1) N_A, M_A are defined and make them into A-bimodules. Let $K = \mathrm{Hom}_A(M_A, N_A)$ the right A-module homomorphisms. It is routine to check, that the following makes K into a left $A \# H$-module:

$$((a \# h)\cdot f)(m) = a \sum (h_{(1)} \cdot f(Sh_{(2)}\cdot m))$$

all $a \in A$, $h \in H$, $f \in K$, $m \in M$.

Now we show that $K^H = \mathrm{Hom}_{A\,\#\,H}({}_AM, {}_AN)$. First, we show for completeness the known fact that $f \in (\mathrm{Hom}_k(M, N))^H$ iff $f \in \mathrm{Hom}_H(M, N)$. Indeed, let $f \in (\mathrm{Hom}_k(M, N))^H$, then

$$\begin{aligned} h\cdot(f(m)) &= \sum h_{(1)}\cdot(f((\varepsilon(h_{(2)}))m)) \\ &= \sum h_{(1)}\cdot f(S(h_{(2)})\, h_{(3)}\cdot m) = \sum \varepsilon(h_{(1)} f(h_{(2)}\cdot m)) \\ &= f(h\cdot m), \end{aligned}$$

so $f \in \mathrm{Hom}_H(M, N)$.

Conversely, if $f \in \mathrm{Hom}_H(M, N)$, $(h\cdot f)(m) = \sum h_{(1)}\cdot(f(S(h_2)\cdot m)) = \sum h_{(1)} S(h_{(2)})\cdot f(m) = \varepsilon(h) f(m)$, so $f \in (\mathrm{Hom}_k(M, N))^N$.

Finally, to prove the claim we show that for $f \in \mathrm{Hom}_H(M, N)$ we have $f \in \mathrm{Hom}(M_A, N_A)$ if and only if $f \in \mathrm{Hom}({}_AM, {}_AN)$.

Indeed, if $f \in \mathrm{Hom}({}_AM, {}_AN)$ then

$$\begin{aligned} f(m \leftharpoonup a) &= f\left(\sum (R^{(2)}\cdot a)(R^{(1)}\cdot m)\right) \\ &= \sum (R^{(2)}\cdot a)\, f(R^{(1)}\cdot m) \\ &= \sum (R^{(2)}\cdot a)(R^{(1)}\cdot (f(m))) = (f(m)) \leftharpoonup a, \end{aligned}$$

that is $f \in \mathrm{Hom}(M_A, N_A)$. The proof of the converse is similar.

If A, B are H-module algebras, $A \otimes B$ is not necessarily an H-module algebra. When (H, R) is quasitriangular $A \otimes B$ can be given such a structure.

PROPOSITION 2.6. *Let (H, R) be a quasitriangular Hopf algebra and let A, B be H-module algebras then*:

(1) *$A \otimes B$ is an H-module algebra, where $(a \otimes b)(c \otimes d) = \sum a(R^{(2)} \cdot c) \otimes (R^{(1)} \cdot b)\, d$. As usual $A \subset A \otimes B$ and $B \subset A \otimes B$ as subalgebras* [Maj].

(2) *If (H, R) is triangular and A and B are H-commutative then so is $A \otimes B$.*

Proof. (2) First we show that the multiplication is associative. Let $a, c, e \in A$ $b, d, f \in B$ then

$$\begin{aligned}
&[(a \otimes b)(c \otimes d)](e \otimes f)\\
&\quad = \sum a(R^{(2)} \cdot c)(r^{(2)} \cdot e) \otimes r^{(1)} \cdot ((R^{(1)} \cdot b)\, d)\, f\\
&\quad = \sum a(R^{(2)} \cdot c)(r^{(2)} \cdot e) \otimes (r^{(1)}_{(1)} R^{(1)} \cdot b)(r^{(1)}_{(2)} \cdot b)\, f.
\end{aligned}$$

Using the formula for $(\Delta \otimes \mathrm{id})(R)$, this equals,

$$= \sum a(R^{(2)} \cdot c)(r^{(2)} s^{(2)} \cdot e) \otimes (r^{(1)} R^{(1)} \cdot b)(s^{(1)} \cdot d)\ f \qquad (*)$$

while

$$\begin{aligned}
&(a \otimes b)[(c \otimes d)(e \otimes f)]\\
&\quad = \sum a r^{(2)} \cdot (c(s^{(2)} \cdot e)) \otimes (r^{(1)} \cdot b)(s^{(1)} \cdot d)\\
&\quad = \sum a(r^{(2)}_{(1)} \cdot c)(r^{(2)}_{(2)} s^{(2)} \cdot e) \otimes (r^{(1)} \cdot b)(s^{(1)} \cdot d).
\end{aligned}$$

Using the formula for $(\mathrm{id} \otimes \Delta)(r)$ this equals $*$, and hence multiplication is associative.

To see that $A \subset A \otimes B$, let $a, a' \in A$ then

$$\begin{aligned}
(a \otimes 1)(a' \otimes 1) &= \sum a R^{(2)} \cdot a' \otimes (R^{(1)} \cdot 1)\\
&= \sum a R^{(2)} \cdot a' \otimes \varepsilon(R^{(1)})\\
&= a\left(\sum (r^{(2)} \varepsilon(R^{(1)})) \cdot a'\right) \otimes 1\\
&= aa' \otimes 1.
\end{aligned}$$

To show H acts on $A\otimes B$ let $a, c\in A$, $b, d\in B$ $h\in H$ then

$$\begin{aligned} &h\cdot[(a\otimes b)(c\otimes d)] \\ &\quad=h\cdot\sum a(R^{(2)}\cdot c)\otimes(R'\cdot b)d \\ &\quad=\sum h_{(1)}\cdot(aR^{(2)}\cdot c)\otimes h_{(2)}\cdot((R^{(1)}\cdot b)d) \\ &\quad=\sum(h_{(1)}\cdot a)(h_{(2)}R^{(2)}\cdot c)\otimes(h_{(3)}R^{(1)}\cdot b)(h_{(4)}\cdot d) \\ &\quad=\sum(h_{(1)}\cdot a)(R^{(2)}h_{(3)}\cdot c)\otimes(R^{1()}h_{(2)}\cdot b)(h_{(4)}\cdot b) \\ &\qquad\qquad(\text{since } \Delta^{\mathrm{cop}}(h)R=R\,\Delta(h)) \\ &\quad=\sum(h_{(1)}\cdot(a\otimes b))(h_{(2)}\cdot(c\otimes d)). \end{aligned}$$

(2) Using the formulas for $(\mathrm{id}\otimes\Delta)(R)$ and $(\Delta\otimes\mathrm{id})(R)$ we obtain

$$(\Delta\otimes\Delta)(R)=\sum R^{(1)}T^{(1)}\otimes r^{(1)}t^{(1)}\otimes T^{(2)}t^{(2)}\otimes R^{(2)}r^{(2)}. \qquad (**)$$

Let $a, c\in A$ $b, c\in B$ then

$$\begin{aligned} &\sum(R^{(2)}\cdot(c\otimes d))(R^{(1)}\cdot(a\otimes b)) \\ &\quad=\sum(T^{(2)}t^{(2)}\cdot c)(u^{(2)}R^{(1)}T^{(1)}\cdot a) \\ &\qquad\otimes(u^{(1)}R^{(2)}r^{(2)}\cdot d)(r^{(1)}t^{(1)}\cdot b) \qquad \text{by } (**). \end{aligned}$$

If $R^{\tau}=R^{-1}$ then $\sum u^{(2)}R^{(1)}\otimes u^{(1)}R^{(2)}=1\otimes 1$ and hence the above equals

$$\sum(T^{(2)}t^{(2)}\cdot c)(T^{(1)}\cdot a)\otimes(r^{(2)}\cdot d)(r^{(1)}t^{(1)}\cdot b).$$

Since A and B are H-commutative this equals

$$=\sum a(t^{(2)}\cdot c)\otimes(t^{(1)}\cdot b)\,d=(a\otimes b)(c\otimes d).$$

Thus $A\otimes B$ is H-commutative.

We end by defining the appropriate notion of centralizers. When H is a cocommutative Hopf algebra acting on an algebra A it was shown [C] that the center of A is H-stable, however, this is not true in general. When (H, R) is quasitriangular we define an H-center, or more generally H-centralizers, which are H-stable.

DEFINITION 2.7. Let (H, R) be a quasitriangular Hopf algebra acting on A. Let $S \subset A$ and define the H-centralizers:

$$C^l(S) = \left\{ a \in A \mid as = \sum (R^{(2)} \cdot s)(R^{(1)} \cdot a), \text{ all } s \in S \right\}$$

$$C^r(S) = \left\{ a \in A \mid as = \sum (R^{(1)} \cdot a)(R^{(2)} \cdot s), \text{ all } s \in S \right\}.$$

When $S = A$ denote $C^l(S) = Z^l_H(A)$, $C^r(A) = Z^r_H(A)$ and let $Z_H(A) = Z^l(H) \cap Z^r(A)$.

PROPOSITION 2.8. *Let $S \subset A$ be H-stable then*

(1) *$C^l(S)$ and $C^r(S)$ are H-stable subalgebras of A.*

(2) *If A satisfies property* $(*)$:

$(*)$ $\sum (R^{(2)} \cdot x)(R^{(1)} \cdot y) = \sum (SR^{(1)} \cdot x)(R^{(2)} \cdot y)$, *all* $x, y \in A$ *then* $C^l(S) = C^r(S)$.

In particular, if (H, R) is triangular then $Z_H(A) = Z^l(H) = Z^r(H)$.

(3) $(C^l(S))^H = (C^r(S))^H = C_A(S) \cap A^H$ *(where $C_A(S)$ denotes the usual centralizer).*

Proof. (1) Let $a, b \in C^l(S)$, $x \in S$, then

$$\begin{aligned}(ab)x &= a \sum (R^{(2)} \cdot x)(R^{(1)} \cdot b) = \sum (r^{(2)} R^{(2)} \cdot x)(r^{(1)} \cdot a)(R' \cdot b) \\ &= \sum (r^{(2)} \cdot x)(r^{(1)}_{(1)} \cdot a)(r^{(1)}_{(2)} \cdot b) \\ &= \sum (r^{(2)} \cdot x)\, r^{(1)} \cdot (ab).\end{aligned}$$

Hence $ab \in C^l(S)$. Thus $C^l(S)$ is a subalgebra of A. To show it is H-stable, let $h \in H$, $a \in C^l(S)$, $x \in S$ then by (0.1)

$$\begin{aligned}(h \cdot a)x &= \sum h_{(1)}(a(S(h_{(2)}) \cdot x) \\ &= \sum h_{(1)}((R^{(2)} S(h_{(2)}) \cdot x)(R^{(1)} \cdot a) \\ &= \sum (h_{(1)} R^{(2)} S(h_{(3)}) \cdot x)(h_{(2)} R^{(1)} \cdot a) \\ &= \sum (R^{(2)} h_{(2)} S(h_{(3)}) \cdot x)(R^{(1)} h_{(1)} \cdot a) \\ &= \sum (R^{(2)} \cdot x)(R^{(1)} \cdot (h \cdot a)).\end{aligned}$$

The right-handed analogue is proved similarly.

(2) Let $a \in C^l(S)$, $x \in S$ then

$$xa = \sum (R^{(2)}r^{(2)} \cdot x)(R^{(1)}S(r^{(1)}) \cdot a)$$

$$= \sum (S(r^{(1)}) \cdot a)(r^{(2)} \cdot x) \qquad \text{(by part (1))}.$$

Hence if A satisfies the given condition we obtain $xa = \sum (R^2 \cdot a)(R^{(1)} \cdot x)$ that is $a \in C^r(S)$. The reverse inclusion is proved similarly.

(3) follows directly from the fact that $\sum R^{(1)} \varepsilon(R^{(2)}) = 1$.

Some Remarks. Condition (*) appropriately translated for the special case $H = (kG)^* \# kG$ is called by [Ha] "dyslectic." It is easy to see that if A is quantum commutative with respect to any (H, R) then it satisfies (*). Also if (H, R) is triangular then any H-module satisfies (*). Moreover, if H is cocommutative (with $R = 1 \otimes 1$) then the H-centralizers coincide with the usual ones, in particular $Z_H(A) = Z(A)$.

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